

Relations among Besov-Type Spaces, Triebel-Lizorkin-Type Spaces and Generalized Carleson Measure Spaces

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Abstract In this paper, the authors construct some counterexamples to show that the generalized Carleson measure space and the Triebel-Lizorkin-type space are not equivalent for certain parameters, which was claimed to be true in [Taiwanese J. Math. 15 (2011), 919-926]. Moreover, the authors show that for some special parameters, the generalized Carleson measure space, the Triebel-Lizorkin-type space and the Besov-type space coincide with certain Triebel-Lizorkin space, which answers a question posed in Remark 6.11(i) of [Lecture Notes in Mathematics 2005, Springer-Verlag, Berlin, 2010, xi+281 pp.]. In conclusion, the Triebel-Lizorkin-type space and the Besov-type space become the classical Besov spaces, when the fourth parameter is sufficiently large.

Function spaces have been widely used in various areas of analysis such as harmonic analysis and partial differential equations. In recent years, there has been increasing interest in a new family of function spaces, called Q_α spaces with $\alpha \in \mathbb{R}$; see, for example, [1, 2, 13, 14] and their references for a history of these spaces.

On the other hand, the most known general scales of function spaces are the scales of Besov spaces and Triebel-Lizorkin spaces. It is well known that Triebel-Lizorkin spaces $\dot{F}_{p,q}^s$ and $F_{p,q}^s$, and Besov spaces $\dot{B}_{p,q}^s$ and $B_{p,q}^s$ on \mathbb{R}^n respectively domains in \mathbb{R}^n for the full ranges of parameters $s \in \mathbb{R}$ and $p, q \in (0, \infty]$ were introduced between 1959 and 1975; see, for example, [10]. Moreover, it is known that Triebel-Lizorkin spaces cover many well-known classical concrete function spaces such as Hölder-Zygmund spaces, Sobolev spaces, fractional Sobolev spaces (also often referred to as Bessel-potential spaces), Hardy spaces and BMO (\mathbb{R}^n), which have their own history. A comprehensive treatment of these function spaces and their history can be found in Triebel's monographs [11, 12].

Recently, Dafni and Xiao [1] introduced the Hardy-Hausdorff space $HH_{-\alpha}^1(\mathbb{R}^n)$ with $\alpha \in (0, \min\{1, n/2\})$ and proved that these spaces are predual spaces of $Q_\alpha(\mathbb{R}^n)$. It was also asked in [1] whether there exist some relations among $Q_\alpha(\mathbb{R}^n)$, $HH_{-\alpha}^1(\mathbb{R}^n)$ and some classical function spaces such as Besov and Triebel-Lizorkin spaces. To answer this

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question, motivated by the Carleson measure characterization of $Q_\alpha(\mathbb{R}^n)$ spaces in [1], we in [15] introduced the Triebel-Lizorkin-type spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $p \in (1, \infty)$ and $q \in (1, \infty]$ and their preduals, the Triebel-Lizorkin-Hausdorff spaces $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $p \in (1, \infty)$, $q \in [1, \infty)$ and $\tau \in [0, \frac{1}{(\max\{p, q\})'}]$, and proved therein that these spaces contain classical Triebel-Lizorkin spaces, Q spaces $Q_\alpha(\mathbb{R}^n)$ and Hardy-Hausdorff spaces $\text{HH}_{-\alpha}^1(\mathbb{R}^n)$ as special cases.

We in [16] further extended the spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ to all $p \in (0, \infty)$ and $q \in (0, \infty]$. Furthermore, the Besov-type spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $p, q \in (0, \infty]$ and their preduals, the Besov-Hausdorff spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $p, q \in [1, \infty)$, $\max\{p, q\} > 1$ and $\tau \in [0, \frac{1}{(\max\{p, q\})'}]$, were also introduced in [16]. It is easy to see that $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ cover the classical Besov spaces as special cases. Some properties of the spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, including the φ -transform characterizations, Sobolev-type embedding properties and smooth atomic and molecular decompositions of these spaces, were also established in [16].

Recently, Lin and Wang in [5] claimed that the Triebel-Lizorkin-type space $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is equivalent to their generalized Carleson measure space $CMO_{\tau q+1-q/p}^{s,q}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $p, q \in (0, \infty)$. We denote the index α in [5] by s here as in [15, 16] in accord with the classical Triebel-Lizorkin spaces when $\tau = 0$. However, in this paper, we first present some counterexamples to show that this is not true when $\tau \in [0, 1/p)$ (see Proposition 4 below). Moreover, by a totally different approach from [5] which may be problematic (see Remark 3 below), we prove that for all $p \in (0, \infty]$, $q \in (0, \infty)$ and $\tau \in (1/p, \infty)$, or $q = \infty$ and $\tau \in [1/p, \infty)$, the Triebel-Lizorkin-type space $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ ($p < \infty$) and the Besov-type space $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ are just the classical Triebel-Lizorkin space $\dot{F}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)$ (see Theorem 1 below), which further implies that for all $s \in \mathbb{R}$, $q \in (0, \infty)$ and $r \in (1, \infty)$, the generalized Carleson measure space $CMO_r^{s,q}(\mathbb{R}^n) = \dot{F}_{q,q}^{s,r/q}(\mathbb{R}^n) = \dot{F}_{\infty,\infty}^{s+n(r-1)/q}(\mathbb{R}^n)$ with equivalent norms (see Corollary 3 below). As a consequence, we see that for all $s \in \mathbb{R}$, $p \in (0, \infty)$, $q \in (0, \infty)$ and $\tau \in (1/p, \infty)$ or $q = \infty$ and $\tau \in [1/p, \infty)$, $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) = \dot{F}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n) = CMO_{\tau q+1-q/p}^{s,q}(\mathbb{R}^n)$ with equivalent norms; see Corollary 4(i) below. Thus, even in this case, Corollary 4 also improves the main results in [5]; see Remark 5 below. Also, as a direct consequence of the main result (Theorem 1 below) of this paper, we know that for all $s \in \mathbb{R}$ and $p \in (0, \infty]$, $\dot{B}_{p,\infty}^{s,1/p}(\mathbb{R}^n) = \dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$ with equivalent norms, which is sharp in the sense of Remark 4 below. Moreover, all results obtained in this paper have inhomogeneous versions and we only explicitly state the inhomogeneous version of Theorem 1 at the end of this paper for similarity; see Theorem 2 below. We remark that Theorem 2 below answers a question posed in [17, p. 168, Remark 6.11(i)]; see Remark 6 below.

To recall the notions of $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, we need some notation. Let $\mathcal{S}(\mathbb{R}^n)$ be the set of all *Schwartz functions* on \mathbb{R}^n endowed with the usual topology and $\mathcal{S}'(\mathbb{R}^n)$ its *topology dual*, namely, the space of all bounded linear functionals on $\mathcal{S}(\mathbb{R}^n)$ endowed with the weak $*$ -topology. Following Triebel [10], we set

$$\mathcal{S}_\infty(\mathbb{R}^n) \equiv \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi(x) x^\gamma dx = 0 \text{ for all multi-indices } \gamma \in (\mathbb{N} \cup \{0\})^n \right\}$$

and consider $\mathcal{S}_\infty(\mathbb{R}^n)$ as a subspace of $\mathcal{S}(\mathbb{R}^n)$, including the topology. Use $\mathcal{S}'_\infty(\mathbb{R}^n)$ to denote the *topological dual space* of $\mathcal{S}_\infty(\mathbb{R}^n)$, namely, the set of all bounded linear functionals on $\mathcal{S}_\infty(\mathbb{R}^n)$. Let $\mathcal{P}(\mathbb{R}^n)$ be the set of all *polynomials* on \mathbb{R}^n . It is well known that $\mathcal{S}'_\infty(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ as topological spaces; see, for example, [17, Proposition 8.1].

Let \mathcal{Q} be the set of all *dyadic cubes* in \mathbb{R}^n , namely,

$$\mathcal{Q} \equiv \{Q_{jk} \equiv 2^{-j}([0, 1]^n + k) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}.$$

For any $Q = Q_{jk} \in \mathcal{Q}$, let $x_Q \equiv 2^{-j}k$, $\ell(Q)$ be the *side-length* of Q , $j_Q \equiv -\log_2 \ell(Q)$, and χ_Q be the *characteristic function* of Q . For all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, Schwartz functions φ and $Q \in \mathcal{Q}$, let $\varphi_j(x) \equiv 2^{jn}\varphi(2^jx)$ and $\varphi_Q(x) \equiv |Q|^{-1/2}\varphi((x - x_Q)/\ell(Q))$. Denote by $\widehat{\varphi}$ the *Fourier transform* of $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$(1) \quad \text{supp } \widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\} \quad \text{and} \quad |\widehat{\varphi}(\xi)| \geq C > 0 \quad \text{when} \quad 3/5 \leq |\xi| \leq 5/3.$$

We now recall the notions of the Triebel-Lizorkin-type space $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and the Besov-type space $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [15, 16] as follows.

Definition 1. Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $q \in (0, \infty]$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (1).

(i) The *Triebel-Lizorkin-type space* $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p \in (0, \infty)$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^{\infty} 2^{jsq} |\varphi_j * f(x)|^q \right]^{p/q} dx \right\}^{1/p} < \infty$$

with the usual modification made when $q = \infty$

(ii) The *Besov-type space* $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p \in (0, \infty]$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} 2^{jsq} \left[\int_P |\varphi_j * f(x)|^p dx \right]^{q/p} \right\}^{1/q} < \infty$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

We also recall the Triebel-Lizorkin-Morrey space $\dot{\mathcal{E}}_{u,p,q}^s(\mathbb{R}^n)$ and the Besov-Morrey space $\dot{\mathcal{N}}_{u,p,q}^s(\mathbb{R}^n)$ introduced in [9, 7] as follows.

Definition 2. Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$, $q \in (0, \infty]$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (1). The *Triebel-Lizorkin-Morrey space* $\dot{\mathcal{E}}_{u,p,q}^s(\mathbb{R}^n)$ and the *Besov-Morrey space* $\dot{\mathcal{N}}_{u,p,q}^s(\mathbb{R}^n)$ are defined, respectively, to be the spaces of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{\mathcal{E}}_{u,p,q}^s(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{Q}} |P|^{\frac{1}{u}-\frac{1}{p}} \left\{ \int_P \left[\sum_{j=-\infty}^{\infty} 2^{jsq} |\varphi_j * f(x)|^q \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} < \infty$$

and

$$\|f\|_{\dot{N}_{u,p,q}^s(\mathbb{R}^n)} \equiv \left\{ \sum_{j=-\infty}^{\infty} \sup_{P \in \mathcal{Q}} |P|^{\frac{q}{u}-\frac{q}{p}} \left[\int_P 2^{jsp} |\varphi_j * f(x)|^p dx \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} < \infty$$

with the usual modifications made when $q = \infty$.

Some known relations among Triebel-Lizorkin spaces, Besov spaces, Triebel-Lizorkin-type spaces, Besov-type spaces, Triebel-Lizorkin-Morrey spaces, Besov-Morrey spaces and Q spaces are summarized as follows. We refer to [3], [16, Propositions 3.1 and 3.2] and [8] for more details.

Proposition 1. *Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $q \in (0, \infty]$. Then*

- (i) $\dot{F}_{p,q}^{s,0}(\mathbb{R}^n) = \dot{F}_{p,q}^s(\mathbb{R}^n)$ for all $p \in (0, \infty)$ and $\dot{B}_{p,q}^{s,0}(\mathbb{R}^n) = \dot{B}_{p,q}^s(\mathbb{R}^n)$ for all $p \in (0, \infty]$.
- (ii) For all $p \in (0, \infty)$, $\dot{F}_{p,q}^{s,1/p}(\mathbb{R}^n) = \dot{F}_{\infty,q}^s(\mathbb{R}^n)$ ([3, Corollary 5.7]).
- (iii) For all $p, q \in (0, \infty)$, $\dot{B}_{\infty,q}^s(\mathbb{R}^n)$ is a proper subspace of $\dot{B}_{p,q}^{s,1/p}(\mathbb{R}^n)$; for all $q \in (0, \infty)$, $\dot{B}_{p,q}^{s,1/p}(\mathbb{R}^n) \subseteq \dot{B}_{q,q}^{s,1/q}(\mathbb{R}^n)$ if $p \geq q$ and $\dot{B}_{q,q}^{s,1/q}(\mathbb{R}^n) \subseteq \dot{B}_{p,q}^{s,1/p}(\mathbb{R}^n)$ if $p \leq q$ ([16, Proposition 3.2]).
- (iv) If $\tau < 0$, then $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) = \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n) = \mathcal{P}(\mathbb{R}^n)$.
- (v) $Q_\alpha(\mathbb{R}^n) = \dot{F}_{2,2}^{\alpha,1/2-\alpha/n}(\mathbb{R}^n)$ for all $\alpha \in (0, \min\{1, n/2\})$ ([15, Corollary 3.1]).
- (vi) For all $0 < p \leq u < \infty$ and $q \in (0, \infty]$, $\dot{\mathcal{E}}_{u,p,q}^s(\mathbb{R}^n) = \dot{F}_{p,q}^{s,1/p-1/u}(\mathbb{R}^n)$ and $\dot{N}_{u,p,\infty}^s(\mathbb{R}^n) = \dot{B}_{p,\infty}^{s,1/p-1/u}(\mathbb{R}^n)$ with equivalent norms; for all $0 < p < u < \infty$ and $q \in (0, \infty)$, $\dot{N}_{u,p,q}^s(\mathbb{R}^n) \subsetneq \dot{B}_{p,q}^{s,1/p-1/u}(\mathbb{R}^n)$ ([8, Theorem 1.1]).

The corresponding sequence spaces, $\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)$, of the spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, were also introduced in [16].

Definition 3. Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $q \in (0, \infty]$. The *sequences spaces* $\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p \in (0, \infty)$ and $\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p \in (0, \infty]$ are defined, respectively, to be the space of all sequences $t \equiv \{t_Q\}_{Q \in \mathcal{Q}} \subset \mathbb{C}$ such that $\|t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$ and $\|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$, where

$$\|t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left(\sum_{Q \subset P} [|Q|^{-s/n-1/2} |t_Q| \chi_Q(x)]^q \right)^{p/q} dx \right\}^{1/p}$$

and

$$\|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left(\int_P \left[\sum_{\substack{Q \subset P \\ \ell(Q)=2^{-j}}} |Q|^{-s/n-1/2} |t_Q| \chi_Q(x) \right]^p dx \right)^{q/p} \right\}^{1/q}$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

Via the Calderón reproducing formula, we in [16] established the φ -transform characterizations of the spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, which implies the following conclusions.

Proposition 2. Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $q \in (0, \infty]$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (1).

(i) For all $p \in (0, \infty)$, $f \in \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and

$$\sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left(\sum_{Q \subset P} \left[|Q|^{-s/n-1/2} |\langle f, \varphi_Q \rangle| \chi_Q(x) \right]^q \right)^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} < \infty.$$

Moreover, $\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)}$ is equivalent to $\|\{\langle f, \varphi_Q \rangle\}_{Q \in \mathcal{Q}}\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)}$ with equivalent constants independent of f .

(ii) For all $p \in (0, \infty]$, $f \in \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and

$$\sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left(\int_P \left[\sum_{\substack{Q \subset P \\ \ell(Q)=2^{-j}}} |Q|^{-s/n-1/2} |\langle f, \varphi_Q \rangle| \chi_Q(x) \right]^p dx \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} < \infty.$$

Moreover, $\|f\|_{\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)}$ is equivalent to $\|\{\langle f, \varphi_Q \rangle\}_{Q \in \mathcal{Q}}\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)}$ with equivalent constants independent of f .

Remark 1. In lines 7 through 11 of [5, p.921], Lin and Wang said that the Triebel-Lizorkin-type space $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $s, \tau \in \mathbb{R}$, $p \in (1, \infty)$ and $q \in (1, \infty]$ was defined in [15] as the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \left\| \{\langle f, \varphi_Q \rangle\}_{Q \in \mathcal{Q}} \right\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty.$$

However, these spaces in [15] were defined as in Definition 1. Moreover, since we did not establish the φ -transform characterization of these spaces in [15], we did not introduce the space $\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)$ of sequences in [15]. Thus, Proposition 2(i) is not included in [15]. However, we do deduce Proposition 2 from the φ -transform characterizations of $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ obtained in a later paper [16].

The generalized Carleson measure space $CMO_r^{s,q}(\mathbb{R}^n)$ for $s, r \in \mathbb{R}$ and $q \in (0, \infty]$ and the space $\dot{B}BMO_p^{s,q}(\mathbb{R}^n)$ for $s \in \mathbb{R}$ and $p, q \in (0, \infty]$ were introduced, respectively, by Lin and Wang in [5] and [4].

Definition 4. Let $s \in \mathbb{R}$ and $q \in (0, \infty]$.

(i) If $r \in \mathbb{R}$, then the *generalized Carleson measure space* $CMO_r^{s,q}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{CMO_r^{s,q}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{Q}} \left\{ |P|^{-r} \int_P \sum_{Q \subset P} \left[|Q|^{-s/n-1/2} |\langle f, \varphi_Q \rangle| \chi_Q(x) \right]^q dx \right\}^{1/q} < \infty$$

with the usual modification made when $q = \infty$.

(ii) If $p \in (0, \infty]$, the space $\dot{B}BMO_p^{s,q}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $\|f\|_{\dot{B}BMO_p^{s,q}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{B}BMO_p^{s,q}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{Q}} \left\{ \sum_{v=j_P}^{\infty} \left[\frac{1}{|P|} \sum_{\substack{Q \subset P \\ \ell(Q)=2^{-v}}} \left(|Q|^{-s/n-1/2+1/p} |\langle f, \varphi_Q \rangle| \right)^p \right]^{q/p} \right\}^{1/q}$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

Remark 2. (i) In [5], Lin and Wang claimed that the generalized Carleson measure space $CMO_r^{s,q}(\mathbb{R}^n)$ was first introduced by themselves in [6].

(ii) As was mentioned in [16], the space $\dot{B}BMO_p^{s,q}(\mathbb{R}^n)$ was introduced in [4] which was the only preprint we had from Lin and Wang when our paper [16] was being written. In [16, p. 463], we even showed that $\dot{B}BMO_p^{s,q}(\mathbb{R}^n)$ is a special case of Besov-type spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$. The spaces $\dot{B}BMO_p^{s,q}(\mathbb{R}^n)$ and $CMO_r^{s,q}(\mathbb{R}^n)$ do obviously not coincide; see Proposition 3 below.

From Propositions 1 and 2, and Definition 4, it is easy to deduce that the spaces $CMO_r^{s,q}(\mathbb{R}^n)$ and $\dot{B}BMO_p^{s,q}(\mathbb{R}^n)$ are, respectively, special cases of Triebel-Lizorkin-type spaces, Triebel-Lizorkin-Morrey spaces and Besov-type spaces as follows; see also [16, p. 463] and [5, p. 921].

Proposition 3. *Let $s \in \mathbb{R}$ and $q \in (0, \infty]$.*

(i) *For all $r \in [0, \infty)$ and $q \in (0, \infty]$, $\dot{F}_{q,q}^{s,\tau/q}(\mathbb{R}^n) = CMO_r^{s,q}(\mathbb{R}^n)$ with equivalent norms. In particular, when $r \in (0, 1)$, $\dot{\mathcal{E}}_{\frac{q}{1-r}, q, q}^s(\mathbb{R}^n) = CMO_r^{s,q}(\mathbb{R}^n)$.*

(ii) *For all $p \in (0, \infty]$, $\dot{B}_{p,q}^{s,1/p}(\mathbb{R}^n) = \dot{B}BMO_p^{s,q}(\mathbb{R}^n)$ with equivalent norms.*

The following is just [5, Theorem 1] with α replaced by s , which is the main result of [5].

Theorem A. *Let $s, \tau \in \mathbb{R}$ and $p, q \in (0, \infty)$. Then*

$$\|\{\langle f, \varphi_Q \rangle\}_{Q \in \mathcal{Q}}\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} \sim \|\{\langle f, \varphi_Q \rangle\}_{Q \in \mathcal{Q}}\|_{\dot{f}_{q,q}^{s,\tau+1/q-1/p}(\mathbb{R}^n)}.$$

The following corollary is immediately deduced from Theorem A, which is just [5, Corollary 6] with α replaced by s .

Corollary A. *Let $s, \tau \in \mathbb{R}$ and $p, q \in (0, \infty)$. Then $\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \sim \|f\|_{CMO_{\tau q+1-q/p}^{s,q}(\mathbb{R}^n)}$.*

Moreover, Theorem A is a direct consequence of the following Theorem B, which is [5, Theorem 2] with α replaced by s .

Theorem B. *Let $s, \tau \in \mathbb{R}$ and $p, q \in (0, \infty)$. Then $\|t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} \sim \|t\|_{\dot{f}_{q,q}^{s,\tau+1/q-1/p}(\mathbb{R}^n)}$.*

Indeed, Theorems A and B and Corollary A when $p = q$ are obvious, and when $\tau = 1/p$ are just [3, Corollary 5.7]; see also Proposition 1(ii). However, it seems that Theorems A and B and Corollary A may be not true for some parameters.

Remark 3. It seems that there exist two gaps in the proof of Theorem B in [5]. For the convenience of the reader, in this remark, we use the same notation as in pages 922 and 923, and page 925 of [5].

First, as in [5, p.922], for all $\alpha \in \mathbb{R}$, dyadic cubes P , sequences $\mathbf{s} = \{s_Q\}_{Q \in \mathcal{Q}}$ and $x \in \mathbb{R}^n$, let

$$G_P^{\alpha, \tau, q}(\mathbf{s})(x) \equiv |P|^{-\tau+1/q} \left\{ \sum_{Q \subset P} \left[|Q|^{-\alpha/n-1/2} |s_Q| \chi_Q(x) \right]^q \right\}^{1/q},$$

$$m^{\alpha, \tau, q}(\mathbf{s})(x) \equiv \sup_{P \in \mathcal{Q}} \inf \{ \varepsilon : |\{x \in P : G_P^{\alpha, \tau, q}(\mathbf{s})(x) > \varepsilon\}| < |P|/4 \}$$

and

$$v(x) \equiv \inf \{ j \in \mathbb{Z} : G_{P_j}^{\alpha, \tau, q}(\mathbf{s})(x) \leq m^{\alpha, \tau, q}(\mathbf{s})(x), \ell(P_j) = 2^{-j} \}.$$

The definition of $v(x)$ is problematic. It seems that the infimum should be taken over all dyadic cubes P containing x ; otherwise $v(x) \equiv -\infty$.

Even if this change is made, it is not clear whether $G_P^{\alpha, \tau, q}(\mathbf{s})(x)$ is monotonic on P , that is, $G_{P_1}^{\alpha, \tau, q}(\mathbf{s})(x) \leq G_{P_2}^{\alpha, \tau, q}(\mathbf{s})(x)$ when $P_1 \subset P_2$ for all $x \in \mathbb{R}^n$. Then the first equality [5, p. 923], namely,

$$E_Q \equiv \{x \in Q : 2^{-v(x)} \geq \ell(Q)\} = \{x \in Q : G_Q^{\alpha, \tau, q}(\mathbf{s})(x) \leq m^{\alpha, \tau, q}(\mathbf{s})(x)\},$$

is problematic. More precisely, the embedding

$$\{x \in Q : 2^{-v(x)} \geq \ell(Q)\} \subset \{x \in Q : G_Q^{\alpha, \tau, q}(\mathbf{s})(x) \leq m^{\alpha, \tau, q}(\mathbf{s})(x)\}$$

may be not true. So, all the proofs break down here.

Second, for all sequences $\mathbf{t} = \{t_Q\}_Q \in \dot{F}_{p,q}^{\alpha, \tau}$, let $\mathcal{Q}(\mathbf{t})$ be the collection of all dyadic cubes Q so that $t_Q \neq 0$ and enumerated as $\mathcal{Q}(\mathbf{t}) \equiv \{P_1, P_2, P_3, \dots\}$. It was claimed in [5, p.925] that \mathbf{t}_m converges to \mathbf{t} in $\dot{F}_{p,q}^{\alpha, \tau}$ as $m \rightarrow \infty$, where \mathbf{t}_m is a sequence containing n non-zero elements of \mathbf{t} , namely, $\mathbf{t}_m \equiv \{(t_m)_Q\}_{Q \in \mathcal{Q}}$ is defined by setting $(t_m)_Q \equiv t_Q$ if $Q \in \{P_1, \dots, P_m\}$, otherwise $(t_m)_Q \equiv 0$ (We replace n in [5, p. 925] by m here to distinguish the dimension of \mathbb{R}^n). However, this may also not be true when $\tau > 0$. For example, when $\alpha = 0$, $p = q = 2$ and $\tau = 1/2$, then $\dot{f}_{2,2}^{0,1/2}$ ($= \dot{f}_{\infty,2}^0$) is the corresponding sequence space of $\text{BMO}(\mathbb{R}^n)$. If $\mathbf{t}_m \rightarrow \mathbf{t}$ in $\dot{f}_{2,2}^{0,1/2}$, applying the φ -transform characterization of $\text{BMO}(\mathbb{R}^n)$, we see that $\mathcal{S}_\infty(\mathbb{R}^n)$ is dense in $\text{BMO}(\mathbb{R}^n)$. But, it is well known that this is not the case.

Indeed, Theorems A and B are not true when $\tau \in [0, 1/p)$. To see this, let $\tau \in [0, 1/p)$ and $q \in (p, \infty)$ such that $\tau + 1/q - 1/p < 0$. Then by Proposition 1(iv), the space $\dot{F}_{q,q}^{s, \tau+1/q-1/p}(\mathbb{R}^n) = \mathcal{P}(\mathbb{R}^n)$. However, it was proved in [16, Proposition 3.1] that the space $\dot{F}_{p,q}^{s, \tau}(\mathbb{R}^n)$ when $\tau \in [0, \infty)$ contains $\mathcal{S}_\infty(\mathbb{R}^n)$, which is a contradiction.

The following proposition give a more concrete counterexample to Theorem B. Recall that $\dot{f}_{p,q}^{s, \tau}(\mathbb{R}^n) = \dot{b}_{p,q}^{s, \tau}(\mathbb{R}^n)$ if $p = q \in (0, \infty)$.

Proposition 4. *Let $s \in \mathbb{R}$.*

(i) *For all $p \in (0, \infty)$, if $q \in (p, \infty)$ and $\tau \in (0, 1/p - 1/q]$, or $q = \infty$ and $\tau \in (0, 1/p - 1/q)$, the space $\dot{b}_{q,q}^{s,\tau+1/q-1/p}(\mathbb{R}^n)$ is a proper subspace of $\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)$.*

(ii) *For all $p \in (0, \infty)$, if $q \in (p, \infty)$ and $\tau \in (0, 1/p - 1/q]$, or $q = \infty$ and $\tau \in [0, 1/p - 1/q)$, the space $\dot{b}_{q,q}^{s,\tau+1/q-1/p}(\mathbb{R}^n)$ is a proper subspace of $\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)$.*

Proof. (i) The embedding $\dot{b}_{q,q}^{s,\tau+1/q-1/p}(\mathbb{R}^n) \subset \dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is a direct consequence of Hölder's inequality. We only show that these two spaces are not equivalent in the case that $q \in (p, \infty)$. The proof of the case that $q = \infty$ is similar and we omit the details.

To this end, for all $j \in \mathbb{Z}$, let $R_j \equiv [0, 2^{-j})^n$. Define $t \equiv \{t_Q\}_Q$ by setting $t_Q \equiv |R_j|^{s/n+1/2+\tau-1/p}$ when $Q = R_j$ for some $j \in \mathbb{Z}$, otherwise $t_Q \equiv 0$. Then, by $\tau > 0$, we conclude that

$$\begin{aligned} \|t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} &= \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left(\sum_{Q \subset P} [|Q|^{-s/n-1/2} |t_Q| \chi_Q(x)]^q \right)^{p/q} dx \right\}^{1/p} \\ &= \sup_{k \in \mathbb{Z}} |R_k|^{-\tau} \left\{ \int_{R_k} \left(\sum_{j=k}^{\infty} [|R_j|^{-s/n-1/2} |t_{R_j}| \chi_{R_j}(x)]^q \right)^{p/q} dx \right\}^{1/p} \\ &\leq \sup_{k \in \mathbb{Z}} |R_k|^{-\tau} \left\{ \sum_{j=k}^{\infty} |R_j|^{(\tau-1/p)p} |R_j| \right\}^{1/p} \sim 1, \end{aligned}$$

while from $\tau \leq 1/p - 1/q$, it follows that

$$\begin{aligned} \|t\|_{\dot{b}_{q,q}^{s,\tau+1/q-1/p}(\mathbb{R}^n)} &= \sup_{P \in \mathcal{Q}} |P|^{-(\tau+1/q-1/p)} \left\{ \int_P \sum_{Q \subset P} [|Q|^{-s/n-1/2} |t_Q| \chi_Q(x)]^q dx \right\}^{1/q} \\ &= \sup_{k \in \mathbb{Z}} |R_k|^{-(\tau+1/q-1/p)} \left\{ \int_{R_k} \sum_{j=k}^{\infty} |R_j|^{(\tau-1/p)q} \chi_{R_j}(x) dx \right\}^{1/q} \\ &= \sup_{k \in \mathbb{Z}} |R_k|^{-(\tau+1/q-1/p)} \left\{ \sum_{j=k}^{\infty} |R_j|^{(\tau-1/p)q} |R_j| \right\}^{1/q} = \infty. \end{aligned}$$

Thus, $\|t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)}$ and $\|t\|_{\dot{b}_{q,q}^{s,\tau+1/q-1/p}(\mathbb{R}^n)}$ are not equivalent, which implies that

$$\dot{b}_{q,q}^{s,\tau+1/q-1/p}(\mathbb{R}^n) \subsetneq \dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n).$$

(ii) Similarly, by Hölder's inequality, we see that $\dot{b}_{q,q}^{s,\tau+1/q-1/p}(\mathbb{R}^n) \subset \dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Again, we only show that $\dot{b}_{q,q}^{s,\tau+1/q-1/p}(\mathbb{R}^n) \subsetneq \dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in the case $q \in (p, \infty)$. The proof of the case that $q = \infty$ is similar and we omit the details.

Let t be as in the proof of (i). Then from $\tau \leq 1/p - 1/q$, we infer that $\|t\|_{\dot{b}_{q,q}^{s,\tau+1/q-1/p}(\mathbb{R}^n)} = \infty$. However, by $\tau > 0$, we obtain

$$\begin{aligned} \|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)} &= \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left(\int_P \sum_{\substack{Q \subset P \\ \ell(Q)=2^{-j}}} [|Q|^{-s/n-1/2} |t_Q| \chi_Q(x)]^p dx \right)^{q/p} \right\}^{1/q} \\ &= \sup_{k \in \mathbb{Z}} |R_k|^{-\tau} \left\{ \sum_{j=k}^{\infty} \left(\int_{R_k} [|R_j|^{-s/n-1/2} |t_{R_j}| \chi_{R_j}(x)]^p dx \right)^{q/p} \right\}^{1/q} \\ &\leq \sup_{k \in \mathbb{Z}} |R_k|^{-\tau} \left\{ \sum_{j=k}^{\infty} |R_j|^{\tau q} \right\}^{1/q} \sim 1, \end{aligned}$$

which completes the proof of Proposition 4. \square

When $\tau \in (1/p, \infty)$, we use a totally different approach from the proof of [5, Theorem 2] to obtain the following conclusions, which have independently interest and may be useful in applications.

Theorem 1. *Let $s \in \mathbb{R}$, $q \in (0, \infty]$.*

(i) *For all $p \in (0, \infty)$, $q \in (0, \infty)$ and $\tau \in (1/p, \infty)$, or $q = \infty$ and $\tau \in [1/p, \infty)$,*

$$\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) = \dot{F}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)$$

with equivalent norms.

(ii) *For all $p \in (0, \infty]$, $q \in (0, \infty)$ and $\tau \in (1/p, \infty)$, or $q = \infty$ and $\tau \in [1/p, \infty)$,*

$$\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n) = \dot{B}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)$$

with equivalent norms.

Proof. (i) By the φ -transform characterizations of the spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [16] and the space $\dot{F}_{p,q}^s(\mathbb{R}^n)$ in [3], to prove (i), it suffices to show that $\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n) = \dot{f}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)$ with equivalent norms, where $\dot{f}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)$ is the sequence space of $\dot{F}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)$; see [3].

To see $\|\cdot\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} \sim \|\cdot\|_{\dot{f}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)}$, we recall that for all $t \equiv \{t_Q\}_{Q \in \mathcal{Q}}$,

$$\|t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left(\sum_{Q \subset P} [|Q|^{-s/n-1/2} |t_Q| \chi_Q(x)]^q \right)^{p/q} dx \right\}^{1/p}$$

and

$$\|t\|_{\dot{f}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)} \equiv \sup_{Q \in \mathcal{Q}} |Q|^{-s/n-(\tau-1/p)-1/2} |t_Q|.$$

Obviously, we have $\|t\|_{\dot{f}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)} \leq \|t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)}$ for all sequences $t \equiv \{t_Q\}_{Q \in \mathcal{Q}}$. On the other hand, by the assumption on τ , we conclude that

$$\begin{aligned}
\|t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} &= \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left(\sum_{j=j_P}^{\infty} \sum_{\substack{Q \subset P \\ \ell(Q)=2^{-j}}} \left[|Q|^{-s/n-1/2} |t_Q| \chi_Q(x) \right]^q \right)^{p/q} dx \right\}^{1/p} \\
&\leq \|t\|_{\dot{f}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)} \\
&\quad \times \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^{\infty} \sum_{\substack{Q \subset P \\ \ell(Q)=2^{-j}}} |Q|^{(\tau-1/p)q} \chi_Q(x) \right]^{p/q} dx \right\}^{1/p} \\
&= \|t\|_{\dot{f}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)} \\
&\quad \times \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^{\infty} 2^{-jn(\tau-1/p)q} \chi_P(x) \right]^{p/q} dx \right\}^{1/p} \\
&\lesssim \|t\|_{\dot{f}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)} \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P |P|^{(\tau-1/p)p} dx \right\}^{1/p} \sim \|t\|_{\dot{f}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)},
\end{aligned}$$

which further implies that $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) = \dot{F}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)$ with equivalent norms and completes the proof of (i).

(ii) Similarly, we only need to show that $\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n) = \dot{b}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)$. The inequality $\|\cdot\|_{\dot{b}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)} \leq \|\cdot\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)}$ is trivial. On the other hand, by the assumption on τ , we see that

$$\begin{aligned}
\|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)} &= \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left(\int_P \sum_{\substack{Q \subset P \\ \ell(Q)=2^{-j}}} \left[|Q|^{-s/n-1/2} |t_Q| \chi_Q(x) \right]^p dx \right)^{q/p} \right\}^{1/q} \\
&\leq \|t\|_{\dot{b}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)} \\
&\quad \times \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left[\int_P \sum_{\substack{Q \subset P \\ \ell(Q)=2^{-j}}} |Q|^{(\tau-1/p)p} \chi_Q(x) dx \right]^{q/p} \right\}^{1/q} \\
&= \|t\|_{\dot{b}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)} \\
&\quad \times \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} 2^{-jn(\tau-1/p)q} \left[\int_P \chi_P(x) dx \right]^{q/p} \right\}^{1/q} \sim \|t\|_{\dot{b}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)},
\end{aligned}$$

which completes the proof of Theorem 1. \square

Observe that $\tau + 1/q - 1/p > 1/q$ when $\tau \in (1/p, \infty)$. As a direct consequence of Theorem 1, we have the following conclusions, comparing with [5, Theorem 1] (see also Theorem A).

Corollary 1. *Let $s \in \mathbb{R}$, $q \in (0, \infty]$.*

(i) *For all $p \in (0, \infty)$, $q \in (0, \infty)$ and $\tau \in (1/p, \infty)$, or $q = \infty$ and $\tau \in [1/p, \infty)$,*

$$\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) = \dot{F}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n) = \dot{B}_{q,q}^{s,\tau+1/q-1/p}(\mathbb{R}^n) \left(= \dot{F}_{q,q}^{s,\tau+1/q-1/p}(\mathbb{R}^n) \text{ if } q < \infty \right)$$

with equivalent norms.

(ii) *For all $p \in (0, \infty]$, $q \in (0, \infty)$ and $\tau \in (1/p, \infty)$, or $q = \infty$ and $\tau \in [1/p, \infty)$,*

$$\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n) = \dot{B}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n) = \dot{B}_{q,q}^{s,\tau+1/q-1/p}(\mathbb{R}^n)$$

with equivalent norms.

Another special case of Theorem 1 is the following conclusion, which has independently interest, comparing with Proposition 1(ii).

Corollary 2. *Let $s \in \mathbb{R}$ and $p \in (0, \infty]$. Then $\dot{B}_{p,\infty}^{s,1/p}(\mathbb{R}^n) = \dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$ with equivalent norms.*

Remark 4. We remark that Corollary 2 is sharp in the following sense: for all $s \in \mathbb{R}$ and $p, q \in (0, \infty)$, $\dot{B}_{\infty,q}^s(\mathbb{R}^n) \subsetneq \dot{B}_{p,q}^{s,1/p}(\mathbb{R}^n)$ by Proposition 1(iii). This is totally different from the case of the Triebel-Lizorkin spaces (see Proposition 1(ii) again).

As a direct consequence of Proposition 3 and Theorem 1, we deduce that when $r \in (1, \infty)$, the space $CMO_r^{s,q}(\mathbb{R}^n)$ is essentially the Triebel-Lizorkin space.

Corollary 3. *Let $s \in \mathbb{R}$. If $q \in (0, \infty)$ and $r \in (1, \infty)$ or $q = \infty$ and $r \in [1, \infty)$, then*

$$CMO_r^{s,q}(\mathbb{R}^n) = \dot{B}_{q,q}^{s,r/q}(\mathbb{R}^n) = \dot{F}_{\infty,\infty}^{s+n(r-1)/q}(\mathbb{R}^n) \left(= \dot{F}_{q,q}^{s,r/q}(\mathbb{R}^n) \text{ if } q < \infty \right)$$

with equivalent norms.

This corollary can be re-written as follows, which implies that the conclusions of Theorems A and B, and Corollary A are correct when $\tau \in (1/p, \infty)$.

Corollary 4. *Let $s \in \mathbb{R}$.*

(i) *If $p \in (0, \infty)$, $q \in (0, \infty)$ and $\tau \in (1/p, \infty)$ or $q = \infty$ and $\tau \in [1/p, \infty)$, then*

$$\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) = \dot{F}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n) = CMO_{\tau q+1-q/p}^{s,q}(\mathbb{R}^n)$$

with equivalent norms.

(ii) *If $p \in (0, \infty]$, $q \in (0, \infty)$ and $\tau \in (1/p, \infty)$ or $q = \infty$ and $\tau \in [1/p, \infty)$, then*

$$\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n) = \dot{B}_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n) = CMO_{\tau q+1-q/p}^{s,q}(\mathbb{R}^n)$$

with equivalent norms.

Remark 5. Although the assertion that for all $s \in \mathbb{R}$, $p, q \in (0, \infty)$ and $\tau \in (1/p, \infty)$, $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) = CMO_{\tau q+1-q/p}^{s,q}(\mathbb{R}^n)$ with equivalent norms, was claimed in [5, Corollary 6], its proof therein is problematic; see Remark 3.

We point out that Theorem 1 is also true for inhomogeneous Triebel-Lizorkin-type spaces $F_{p,q}^{s,\tau}(\mathbb{R}^n)$ and Besov-type spaces $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ introduced in [17]. Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$(2) \quad \text{supp } \widehat{\Phi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\} \quad \text{and} \quad |\widehat{\Phi}(\xi)| \geq C > 0 \quad \text{if} \quad |\xi| \leq 5/3.$$

The inhomogeneous Triebel-Lizorkin-type space $F_{p,q}^{s,\tau}(\mathbb{R}^n)$ and the inhomogeneous Besov-type space $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [17] were defined as follows.

Definition 5. Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $q \in (0, \infty]$. Let φ be as in (1) and $\varphi_0 \equiv \Phi$ be as in (2).

(i) If $p \in (0, \infty)$, the *inhomogeneous Triebel-Lizorkin-type space* $F_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=\max\{0, j_P\}}^{\infty} 2^{jsq} |\varphi_j * f(x)|^q \right]^{p/q} dx \right\}^{1/p} < \infty$$

with the usual modification made when $q = \infty$.

(ii) If $p \in (0, \infty]$, the *inhomogeneous Besov-type space* $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=\max\{0, j_P\}}^{\infty} 2^{jsq} \left[\int_P |\varphi_j * f(x)|^p dx \right]^{q/p} \right\}^{1/q} < \infty$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

All conclusions of Propositions 1 through 4 have inhomogeneous versions and we omit the details. Moreover, we have the following conclusions, whose proofs are similar to that of Theorem 1. We also omit the details.

Theorem 2. Let $s \in \mathbb{R}$ and $q \in (0, \infty]$.

(i) For all $p \in (0, \infty)$, $q \in (0, \infty)$ and $\tau \in (1/p, \infty)$, or $q = \infty$ and $\tau \in [1/p, \infty)$,

$$F_{p,q}^{s,\tau}(\mathbb{R}^n) = F_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)$$

with equivalent norms.

(ii) For all $p \in (0, \infty]$, $q \in (0, \infty)$ and $\tau \in (1/p, \infty)$, or $q = \infty$ and $\tau \in [1/p, \infty)$,

$$B_{p,q}^{s,\tau}(\mathbb{R}^n) = B_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n)$$

with equivalent norms.

Remark 6. It was asked in [17, p. 168, Remark 6.11(i)] that for which set of parameters p, q, τ , the spaces $F_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ coincide with the Hölder-Zygmund spaces? Some special cases were obtained in [17, p. 167, Theorem 6.9]. Obviously, Theorem 2 above gives a complete answer to this question.

From Theorem 2, we also deduce inhomogeneous versions of all conclusions in Corollaries 1 through 4. We omit the details again.

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